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Difference equations, shift operators and systems over Noetherian factorial domains

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Abstract

In this paper we study a class of operators which act on spaces of sequences taking value on a module over a Noetherian factorial domain. These operators are obtained as linear combinations of the operators that shift the sequences forward and backward. For this reason they are called shift operators. The properties of this class of operators are effectively applied to study difference equations and dynamical systems over rings. © 1997 Elsevier Science B.V.

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1. Introduction and notation

1.1. Difference equations over rings

Let R be a ring and let $r_m, \ldots, r_n \in R$ with $m \leq n \in \mathbb{Z}$. Consider the difference equation

$$r_m x(t+m) + r_{m+1} x(t+m+1) + \cdots + r_n x(t+n) = 0. \tag{1}$$

Does there exist a non-zero solution of (1)? In other words, does there exist a non-zero sequence $x = \{x(t)\}_{t\in\mathbb{Z}} \in R^{\mathbb{Z}}$ such that (1) is satisfied for all $t \in \mathbb{Z}$? More generally, consider the non-homogeneous problem: given a sequence $y = \{y(t)\}_{t\in\mathbb{Z}} \in R^{\mathbb{Z}}$, does there exist a sequence $x = \{x(t)\}_{t\in\mathbb{Z}} \in R^{\mathbb{Z}}$ such that

$$r_m x(t+m) + r_{m+1} x(t+m+1) + \cdots + r_n x(t+n) = y(t) \tag{2}$$

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for all $t \in \mathbb{Z}$? We are interested in characterizing the set of all the solutions of (1) and (2) and also to establish concrete algorithms to construct such solutions. If $R = k$ is a field, things are quite simple: excluding the trivial case in which all the r_j's are equal to 0, we can as well assume that r_m and r_n are non-zero. Then, for every $y \in k^{\mathbb{Z}}$, the set of all the solutions of (2) form an affine k-subspace $\mathscr{S}_y \subseteq k^{\mathbb{Z}}$ of dimension $n - m$. The freedom in the solution corresponds to the fact that we can arbitrarily assign the value of x in $n - m$ consecutive time instants and then solve uniquely in a recursive fashion backward and forward in time using (2). Notice that, as a consequence, there exists a non-zero solution of (1) if and only if $n - m > 0$. A remarkable fact is that S_y depends on y in a finite way, namely

$$y_1, y_2 \in k^{\mathbb{Z}} \; y_{1|[a,b]} = y_{2|[a,b]} \;\Rightarrow\; \mathscr{S}_{y_1|[a+m,b+n]} = \mathscr{S}_{y_2|[a+m,b+n]}, \tag{3}$$

where we have used the symbol $|$ to denote the restriction of sequences to a certain index subset. Similar considerations can clearly be repeated for general integral domains R in the case the leading coefficients r_m and r_n are units. There is another useful way to represent solutions of (2) (in the case r_m and r_n are units) which we now quickly recall for later use. Introduce an auxiliary variable

$$\xi(t) := \begin{pmatrix} x(t) \\ x(t+1) \\ \vdots \\ x(t+L-1) \end{pmatrix}, \tag{4}$$

where $L = n - m$. Define now

$$A := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -r_n^{-1}r_m & -r_n^{-1}r_{m+1} & -r_n^{-1}r_{m+2} & \cdots & -r_n^{-1}r_{n-2} & -r_n^{-1}r_{n-1} \end{pmatrix} \tag{5}$$

$$b := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \qquad C := (1 \;\; 0 \;\; \cdots \;\; 0). \tag{6}$$

The solutions of (2) are then given by

$$\begin{cases} \xi(t+1) = A\xi(t) + by(t-m), \\ x(t) = C\xi(t) \end{cases} \tag{7}$$

as $\xi(0) = \xi_0$ varies in R^L. From (7) we get the closed expression

$$x(t) = C[A^t\xi_0 + \sum_{k=0}^{t-1} A^{t-1-k}by(k-m)], \quad t \in \mathbb{Z}. \tag{8}$$

The difficulty starts when at least one of the leading coefficients r_m, r_n is no longer a unit. A lot of pathologies can then happen even for relatively simple rings. As an example, in the case $R = \mathbb{R}[z]$, consider the equations

$$x(t) - zx(t+1) = 0, \tag{9}$$

$$x(t) + x(t+1) + zx(t+2) = 0, \tag{10}$$

$$x(t) + (z-1)x(t+1) - zx(t+2) = 0. \tag{11}$$

It is easy to check that (9) has only the 0 solution and the same is true for (10) (see Section 2). On the other hand, (11) admits non-trivial solutions: $x(t) = p$ constant where $p \in \mathbb{R}[z]$. The non-homogeneous problems associated with the above difference equations are not easy to solve and only partial results can be obtained. These examples will be taken up in Section 2: in particular, we will show that they do not satisfy finite property (3).

Eqs. (1) and (2) can be expressed in a more compact form. Denote by σ the *backward shift* on $R^{\mathbb{Z}}$ defined by $(\sigma x)(t) := x(t+1)$. Consider the Laurent polynomial $p := \sum_{j=m}^{n} r_j u^j$. It induces an R-homomorphism called a *scalar shift operator*

$$p(\sigma, \sigma^{-1}) : R^{\mathbb{Z}} \to R^{\mathbb{Z}}, \tag{12}$$

$$p(\sigma, \sigma^{-1})x := \sum_{j=m}^{n} r_j(\sigma^j x). \tag{13}$$

Eq. (2) can be written as

$$p(\sigma, \sigma^{-1})x = y. \tag{14}$$

In order to solve previous problems one is then naturally led to study the kernel and the image of scalar shift operators.

A straightforward generalization of this problem can be obtained in the following way. First let us set some more notation. If V is an R-module, denote by $V[u, u^{-1}]$ the $R[u, u^{-1}]$-module of Laurent polynomials with coefficients in V. Let V, W be R-modules and consider the corresponding sequence spaces $V^{\mathbb{Z}}$ and $W^{\mathbb{Z}}$ on which the shift σ (by abuse of notation we always denote it with the same symbol) acts as on $R^{\mathbb{Z}}$. Let now $M = \sum_{j=m}^{n} M_j u^j \in \mathrm{Hom}_R(V, W)[u, u^{-1}]$. It induces an R-homomorphism

$$M(\sigma, \sigma^{-1}) : V^{\mathbb{Z}} \to W^{\mathbb{Z}}, \tag{15}$$

$$M(\sigma, \sigma^{-1})v = \sum_{j=m}^{n} M_j(\sigma^j v). \tag{16}$$

$M(\sigma, \sigma^{-1})$ is said to be a *shift operator*. We can study the equation

$$M(\sigma, \sigma^{-1})v = w, \tag{17}$$

where $w \in W^{\mathbb{Z}}$. In the case $V = R^q$ and $W = R^l$, this clearly corresponds to study solution of a system of l difference equations in q distinct sequence variables.

There are several reasons for studying difference equations over rings. We first recall that it follows from the work of Willems [12, 13] that the theory of linear control systems in discrete time is ultimately the study of kernels and images of shift operators over real or complex vector spaces. On the other hand, in the last decades there has been a growing interest in trying to extend the linear control theory to more general algebraic structures like rings and modules [3, 11]: this, in Willems' framework, leads exactly to the objects of our investigations: shift operators over modules [14]. An especially important case is when the ring is an algebra of functions, since in this case difference equations over such rings can be interpreted as a family of linear difference equations parametrized by certain parameters living on a topological space, a manifold, an algebraic variety. In this case, it is reasonable to ask if they admit solutions which are also parametrized with the same regularity that the coefficients had. Rings of functions considered in this paper will mainly be the ring of polynomials $k[z_1,\dots,z_n]$ and the ring of convergent power series $k\{z_1,\dots,z_n\}$. Another motivation is related to the symbolic dynamics over infinite alphabets. As it will appear more clearly in next subsection, kernels and images of shift operators can be interpreted as dynamical systems with a module structure which have a lot of interesting dynamical properties. In this regard the case of the integers $\mathbb{Z}$ is probably the most interesting: see [5].

1.2. *A dynamical systems point of view*

Let V be a finitely generated R-module equipped with the discrete topology and consider the sequence R-module $V^{\mathbb{Z}}$ equipped with the product topology. The dynamical system $(V^{\mathbb{Z}},\sigma)$ is called the *full R-shift* over the *alphabet* V. More generally, let $\mathscr{B}\subseteq V^{\mathbb{Z}}$ be a (closed) σ-invariant R-submodule. Then, the dynamical system $(\mathscr{B},\sigma_{|\mathscr{B}})$ is called a (closed) *R-shift* over V. For the sake of simplicity, whenever this does not cause confusion, the restriction sign in σ will be dropped. Also, we will refer to $\mathscr{B}$ itself as the R-shift $(\mathscr{B},\sigma)$.

Consider two R-shifts $\mathscr{B}_1$ and $\mathscr{B}_2$. A map $\psi\colon \mathscr{B}_1\to\mathscr{B}_2$ is called an *R-morphism* if ψ is a continuous R-homomorphism and $\psi\circ\sigma=\sigma\circ\psi$. Shift operators are R-morphisms between full R-shifts and it can easily be shown that these are all the possible ones. It thus follows that kernels of shift operators are closed R-shifts. On the other hand, as we will see later on, an image of a shift operator may not be closed. It is, however, an R-shift.

Kernels and images have a sort of duality property: given a kernel R-shift $\mathscr{B}$, it is difficult to construct elements belonging to it, in particular it is difficult to check if $\mathscr{B}\neq\{0\}$. On the other hand, it is easy to check if a given sequence v is in $\mathscr{B}$. The converse is true for image R-shifts: it is difficult to establish if a given sequence is in $\mathscr{B}$ but it is easy to construct elements. For this reason, when one is confronted with the difference equations

$$M(\sigma,\sigma^{-1})v=0, \tag{18}$$

$$M(\sigma,\sigma^{-1})v=w \tag{19}$$

may naturally pose the following problems:

(1) Parametrize the set of solutions of (18).

(2) Check the solvability of (19).

(3) Parametrize the set of solutions of (19).

The answers to the previous questions are connected with two important concepts of systems theory and symbolic dynamics: controllability and finite memory. Let $\mathscr{B} \subseteq V^{\mathbb{Z}}$ be an R-shift. $\mathscr{B}$ is said to be *controllable* [12] if for all $v_1, v_2 \in \mathscr{B}$, there exists $n \in \mathbb{N}$ and $v \in \mathscr{B}$ with

$$v(t) = v_1(t) \ \forall t < 0, \quad (\sigma^n v)(t) = v_2(t) \ \forall t \geq 0. \tag{20}$$

On the other hand, if $I \subseteq \mathbb{Z}$, denote by $\mathscr{B}_{|I}$ the R-module of restrictions of the bi-infinite sequences in $\mathscr{B}$ to I. $\mathscr{B}$ is said to have *memory* $n \in \mathbb{N}$ if

$$v \in V^{\mathbb{Z}} \quad \text{and} \quad v_{|[t,t+n]} \in \mathscr{B}_{|[t,t+n]} \ \forall t \in \mathbb{Z} \Rightarrow v \in \mathscr{B}. \tag{21}$$

$\mathscr{B}$ is said to have *finite memory* (or to be of *finite type*), if it has memory n for some $n \in \mathbb{N}$. In the field case it happens that every closed R-shift has finite memory. This is not true in general, not even for principal ideal domains (PID's) [4]. It can easily be shown that if $\mathscr{B}$ is controllable, then it is topologically transitive as a dynamical system. The converse is also true under the assumption that $\mathscr{B}$ has finite memory.

Consider the first problem. As mentioned above an efficient way to parametrize the set of solutions $\operatorname{Ker} M(\sigma, \sigma^{-1})$ would be to express it as the image of a suitable shift operator. It has been shown in [14] that a closed R-shift can be expressed as the image of a shift operator if and only if it is controllable. It is clear that a kernel is closed but not necessarily controllable. For instance, in the scalar case discussed above we have that a kernel is never controllable, unless it is $\{0\}$ or $R^{\mathbb{Z}}$. Therefore, in order to understand when the solution set of a homogeneous difference equation admits an image representation, we have to characterize the class of kernel R-shifts which are controllable. Also when the set of solutions $\mathscr{B} := \operatorname{Ker} M(\sigma, \sigma^{-1})$ is not controllable, there exists the largest controllable closed R-shift inside $\mathscr{B}$, that is denoted as $\mathscr{B}_c$ and that can be described as the image of a suitable shift operator. We will see in the sequel that in general $\mathscr{B}$ can be written as sum

$$\mathscr{B} = \mathscr{B}_c + \tilde{\mathscr{B}},$$

where $\tilde{\mathscr{B}}$ is a finitely generated free R-shift. Therefore $\mathscr{B}_c$ can be represented as the image of a suitable shift operator, while the elements of $\tilde{\mathscr{B}}$ can be fruitfully described through a generalized initial conditions fashion. The only drawback of this parametrization is that it is not injective in general since the previous sum can not always be found directly, unless R is a field.

Consider now the second problem. This consists in finding an efficient way to decide whether for a certain w there exists a solution or not, or, equivalently, in finding an efficient method for checking if $w \in \operatorname{Im} M(\sigma, \sigma^{-1})$. As mentioned above, expressing $\operatorname{Im} M(\sigma, \sigma^{-1})$ as the kernel of a suitable shift operator would provide this method.

However there are cases in which this is not possible and these occur just when $\operatorname{Im} M(\sigma, \sigma^{-1})$ does not have finite memory. Actually, it is clear that kernels have necessarily finite memory. Also the converse is true: if $\mathscr{B} \subseteq V^{\mathbb{Z}}$ has memory N, consider the R-projection

$$f: V^{N+1} \to V^{N+1}/\mathscr{B}_{|[0,N]} = W. \tag{22}$$

It induces a shift operator

$$\psi: V^{\mathbb{Z}} \to W^{\mathbb{Z}} \tag{23}$$

by

$$(\psi v)(t) := f(v_{|[t,t+N]}) \tag{24}$$

and $\mathscr{B} = \operatorname{Ker} \psi$. It may well happen that W is not free even if V was. See [6] for further discussion on this point.

In the case R is a field, every closed R-shift has finite memory and moreover the image of a shift operator is always closed. Consequently in this case the situation is simple. When R is a ring (different from the field case) images of shift operators are not necessarily closed, in particular they may not be kernels. In these cases it turns out to be very difficult to exploit the structure of these images and we do not know any systematic way to do it. In Section 2 we discuss some techniques for the scalar case (see Examples 1–4). In principle it could even happen that an image is closed but does not have finite memory. We do not have any example of this sort but there is a positive result: if R is a PID, then, if an image is closed, it has finite memory [5]. If an image has finite memory then, by previous considerations it is also a kernel.

Finally, since Eq. (19) is linear, the third problem reduces to solving problem 1 and to finding a particular solution of (19).

1.3. Outline of the contents and extensions

All rings considered in this paper will be commutative Noetherian with identity element. In Section 2 we make a fundamental study of scalar shift operators in the case when R is a commutative Noetherian factorial domain (or unique factorization domain UFD). We establish a correspondence between set- and topological-theoretic properties of such maps and algebraic properties of inducing polynomials. We then pass to show how these results can be fruitfully applied to study scalar difference equations: we have a complete result for the homogeneous case and partial results for the non-homogeneous case. In Section 3 we pass to the matrix case for principal ideal domains (PID's). We first make a fundamental study of shift operators and we then pass to consider systems of difference equations: we establish quite complete results for the homogeneous problem and partial ones for the non-homogeneous one. Finally certain extensions to Noetherian factorial domains which are not PID are also considered.

The assumption that R is a Noetherian factorial domain seems to be crucial in order to obtain a systematic theory as the one developed in this paper for the scalar shift

operators. On the other hand, a systematic extension to the matrix case is unlikely to be possible without the assumption that R is a PID. However in many specific situations it may be possible to have remarkable extensions. The work [6] contains many elements which actually go in this direction: there, the properties of the shift operators are more intimately related to dynamical properties of R-shifts than in this paper.

2. Scalar shift operators

2.1. First results

In this section, R always denotes a commutative Noetherian factorial domain (UFD). Denote by R^* and R_0 the sets of invertible elements and of non-zero elements in R, respectively. A multiplicatively closed subset $S \subseteq R$ ($0 \notin S$, $1 \in S$) is said to be *saturated* if for all $a, b \in R$ we have that $ab \in S \Leftrightarrow a, b \in S$. Let $S \subseteq R$ be a saturated multiplicatively closed set. Define

$$\widetilde{S} := \{a \in R \mid (b \notin R^*,\ b \text{ divides } a) \Rightarrow b \notin S\}. \tag{25}$$

It is easy to see that $\widetilde{S}$ is also a saturated multiplicatively closed set in R.

Consider $R[u, u^{-1}]$. $p = \sum_{i=m}^{n} r_i u^i \in R[u, u^{-1}]$, with $r_m, r_n \in R^*$, is said to be *bimonic*. Denote by S_b the saturated multiplicatively closed subset of all the bimonic polynomials in $R[u, u^{-1}]$. S_b and $\widetilde{S}_b$ will play an important role in the sequel. Other relevant multiplicatively closed subsets in $R[u, u^{-1}]$ are R_0 and $(R[u, u^{-1}])^*$. $\widetilde{R}_0$ consists of the so called *primitive* polynomials. Notice that R_0 is not saturated, while its saturation is $R_0 \cdot (R[u, u^{-1}])^* = \widetilde{\widetilde{R}}_0$.

The following was proved in [5]. For completeness we sketch the proof.

Proposition 1. *Consider $p \in R[u, u^{-1}]$. Then, $p(\sigma, \sigma^{-1})$ is injective if and only if $p \in \widetilde{S}_b$.*

Proof. Suppose that $p \notin \widetilde{S}_b$. Then, there exists a non-unit bimonic polynomial q such that $q \mid p$. It is clear that

$$\operatorname{Ker} p(\sigma, \sigma^{-1}) \supseteq \operatorname{Ker} q(\sigma, \sigma^{-1}) \neq 0. \tag{26}$$

Suppose, conversely, that $p \in \widetilde{S}_b$. Since $p \neq 0$ it is easy to see that the R-shift $\mathscr{B} = \operatorname{Ker} p(\sigma, \sigma^{-1})$ is finitely generated over R. Choose $x_1, \ldots, x_n$ R-generators of $\mathscr{B}$ and let $A \in R^{n \times n}$ be such that

$$\sigma \cdot [x_1 \cdots x_n] = [x_1 \cdots x_n] A. \tag{27}$$

Let g be the characteristic polynomial of the matrix A. Since A is invertible, $g \in S_b$ and it can easily be shown that $\operatorname{Ker} g(\sigma, \sigma^{-1}) \supseteq \mathscr{B}$. It is clear that g and p must be coprime polynomials and so there exist $h, k \in R[u, u^{-1}]$ such that $a = hp + kg \in R_0$. Since $a\mathscr{B} = 0$ we now have that $\mathscr{B} = 0$. □

Define now Σ to be the saturated multiplicatively closed set of Laurent polynomials which induce surjective scalar shift operators. It is easy to see that $S_b \subseteq \Sigma$. As we will see equality does not hold in general. We have the following:

Proposition 2. *Let $p \in R[u,u^{-1}]$. Then, $p(\sigma,\sigma^{-1})$ is an open surjection if and only if $p \in S_b$.*

Proof. Assume that $p \in S_b$ and write $p = \sum_{i=h}^{k} r_i u^i$ with $r_h, r_k \in R^*$. Let $y_n \in R^{\mathbb{Z}}$ such that $y_{n|[-n,n]} = 0$. Clearly, for every $n \in \mathbb{N}$ there exists $x_n \in R^{\mathbb{Z}}$ such that $x_{n|[-n+h,n+k]} = 0$ and $p(\sigma,\sigma^{-1})x_n = y_n$. This proves that $p(\sigma,\sigma^{-1})$ is an open surjection.

Suppose, conversely, that $p(\sigma,\sigma^{-1})$ is an open surjection and write $p = p_1 p_2$, with $p_1 \in \widetilde{S}_b$ and $p_2 \in S_b$. $p_1(\sigma,\sigma^{-1})$ is also open and bijective. Let $\delta \in R^{\mathbb{Z}}$ be defined by $\delta(0) = 1$ and $\delta(t) = 0, \forall t \neq 0$ and let $x \in R^{\mathbb{Z}}$ be such that $p_2(\sigma,\sigma^{-1})x = \delta$. Since the sequences $\sigma^n\delta$, $n \in \mathbb{N}$, and $\sigma^{-n}\delta$, $n \in \mathbb{N}$ both converge to zero, $\sigma^n x$ and $\sigma^{-n}x$ must also converge to zero. This implies that x has finite support which yields $p_1 \in (R[u,u^{-1}])^*$. □

2.2. The semisimple case

We now want to study in further detail the relation between S_b and Σ. In some cases we have equality: this happens when the ring R is semisimple (i.e. the intersection of all the maximal ideals is $\{0\}$). Examples of semisimple rings which we have in mind are the ring of integers $\mathbb{Z}$ and the ring of polynomials over a field $k[z_1,\ldots,z_s]$.

We start with the following:

Lemma 3. *Assume that R is semisimple. Let $p \in R[u,u^{-1}]$ such that $p(\sigma,\sigma^{-1})$ is injective and $\operatorname{Im} p(\sigma,\sigma^{-1}) \supseteq aR^{\mathbb{Z}}$ for some $a \in R_0$. Then, $p \in R_0 \cdot (R[u,u^{-1}])^*$.*

Proof. Write $p = \sum_{i=m}^{n} r_i u^i$. Let m be a maximal ideal in R such that $a \notin m$. Consider the residue field $k := R/m$ and let $\bar{p} = \sum_{i=m}^{n} \bar{r}_i u^i \in k[u,u^{-1}]$ be the quotient projection of p. It can easily be proven that $\bar{p}(\sigma,\sigma^{-1})$ is a bijective scalar shift operator and since k is a field, we have that [12] $\bar{p} = bu^l$, where $b \in k^*$ and $l \in \mathbb{Z}$. Consider now any pair of coefficients r_i, r_j, with $i \neq j$, of p. Then, $r_i r_j$ is in the intersection J of all the maximal ideals of R which do not contain a. Let J' be the intersection of all the maximal ideals of R containing a. Since R is semisimple we have that $J \cdot J' \subseteq J \cap J' = (0)$. Since $(a) \subseteq J'$ and R is a domain, it follows that $J = (0)$. This yields $r_i r_j = 0$ for all pairs $i \neq j$. Hence, only one coefficient in p is non-zero. □

Proposition 4. *Assume that R is semisimple. Then,*

(1) $\widetilde{S}_b \cap \Sigma = (R[u,u^{-1}])^*$. *In particular, every invertible scalar shift operator has a continuous inverse.*

(2) $S_b = \Sigma$. *In particular, every surjective scalar shift operator is open.*

Proof. (1) Immediately follows from Lemma 3 and Proposition 1.

(2) Let $p \in \Sigma$. Factor $p = p_1 p_2$ with $p_1 \in \widetilde{S}_b$ and $p_2 \in S_b$. Clearly, $p_1 \in \widetilde{S}_b \cap \Sigma = (R[u,u^{-1}])^*$. Hence, $p \in S_b$. □

We have another interesting consequence.

Corollary 5. *Assume that R is semisimple and let $p \in R[u,u^{-1}]$. Then, the following conditions are equivalent:*

(1) $\operatorname{Im} p(\sigma,\sigma^{-1}) \supseteq aR^{\mathbb{Z}}$ *for some* $a \in R_0$.

(2) $\operatorname{Im} p(\sigma,\sigma^{-1})$ *is a closed finite memory R-shift.*

(3) $p \in R_0 \cdot S_b$.

Proof. (1)⇒(3) Write $p = p_1 p_2$ with $p_1 \in \widetilde{S}_b$ and $p_2 \in S_b$. Since $\operatorname{Im} p(\sigma,\sigma^{-1}) = \operatorname{Im} p_1(\sigma,\sigma^{-1})$, we can conclude using Lemma 3.

(3)⇒(2) is evident.

(2)⇒(1) Assume that $\mathscr{B}$ has memory L. Write $p = \sum_{i=m}^{n} r_i u^i$ with $r_m, r_n \neq 0$. It follows that

$$\mathscr{B}_{|[0,L]} = \operatorname{Im} \begin{bmatrix} r_m & \cdots & r_n & & \\ & \ddots & & \ddots & \\ & & r_m & \cdots & r_n \end{bmatrix}. \tag{28}$$

Therefore,

$$\mathscr{B}_{|[0,L]} \supseteq r_m^{L+1} R^{\mathbb{Z}}_{|[0,L]} \tag{29}$$

and, since $\mathscr{B}$ has memory L,

$$\mathscr{B} \supseteq r_m^{L+1} R^{\mathbb{Z}}. \quad \square \tag{30}$$

2.3. The complete case

Quite different results can be obtained for complete rings. Let $m \leq R$ be a maximal ideal. Assume that R is complete with respect to the m-adic topology τ. Consider the ideal $\tilde{m} := m[u,u^{-1}]$ in the ring $R[u,u^{-1}]$. Denote by $\widehat{R[u,u^{-1}]}$ the $\tilde{m}$-completion of $R[u,u^{-1}]$. We can think, in the standard way,

$$\widehat{R[u,u^{-1}]} \hookrightarrow \prod_{n\geq 0} (R/m^n R)[u,u^{-1}]. \tag{31}$$

Proposition 6. *Assume that $R^{\mathbb{Z}}$ is equipped with the product topology τ^{∞} (each factor R is equipped with the topology τ). The τ^{∞}-continuous R-homomorphisms of $R^{\mathbb{Z}}$ which commute with σ are in bijective correspondence with $\widehat{R[u,u^{-1}]}$ in the following way: to $p = \{p_n\} \in \widehat{R[u,u^{-1}]}$ with $p_n \in (R/m^n R)[u,u^{-1}]$, we associate*

$$p(\sigma,\sigma^{-1}) : R^{\mathbb{Z}} \to R^{\mathbb{Z}} \tag{32}$$

defined by

$$(p(\sigma,\sigma^{-1})v)(t) := a, \tag{33}$$

where $a \in R$ *is the unique element such that* $\{a \bmod(m^n)\} = \{(p_n(\sigma,\sigma^{-1})v)(t)\}$ *for all* $n \in \mathbb{N}$.

Proof. It is straightforward to verify that $p(\sigma,\sigma^{-1})$ defined by (33) is continuous. On the other hand, given a τ^∞-continuous R-homomorphism $\phi: R^{\mathbb{Z}} \to R^{\mathbb{Z}}$ which commutes with σ, we can consider the induced

$$\phi_n: (R/m^nR)^{\mathbb{Z}} \to (R/m^nR)^{\mathbb{Z}}. \tag{34}$$

Since the quotient topology on R/m^nR is discrete and ϕ_n is a continuous R/mR-homomorphism which commutes with σ, there exist $p_n \in (R/m^nR)[u,u^{-1}]$ such that $\phi_n = p_n(\sigma,\sigma^{-1})$. It is clear that, if we denote $p = \{p_n\}$, we have that $\phi = p(\sigma,\sigma^{-1})$. □

Proposition 7. *Let* $m \leq R$ *be a maximal ideal and assume that* R *is complete in the* m*-adic topology* τ. *Then,*

(1) $\widetilde{S}_b \cap \Sigma = (\widehat{R[u,u^{-1}]})^* \cap R[u,u^{-1}]$. *This shows that every invertible scalar shift operator has an inverse which is continuous in the* τ^∞*-topology.*

(2) $\Sigma = ((\widehat{R[u,u^{-1}]})^* \cap R[u,u^{-1}]) \cdot S_b$.

Proof. (1) $\supseteq$ follows from Propositions 1 and 6. Conversely, let $p \in \widetilde{S}_b \cap \Sigma$. It follows again from Propositions 1 and 6, that $p(\sigma,\sigma^{-1})$ is τ^∞-continuous and invertible. Since R with the topology τ is inverse limit of Artinian discrete rings, it follows that R is strictly linearly compact [10]. Then, also $R^{\mathbb{Z}}$ is strictly linearly compact and, consequently, $p(\sigma,\sigma^{-1})$ admits a τ^∞-continuous inverse. This yields $p \in (\widehat{R[u,u^{-1}]})^* \cap R[u,u^{-1}]$.

(2) As (2) of Proposition 4. □

Remark. It is easy to see that $p = \{p_n\} \in \widehat{R[u,u^{-1}]}$ is a unit if and only if p_1 is a unit of $(R/m)[u,u^{-1}]$. Let $\alpha \in m \setminus \{0\}$. Clearly, $1 + \alpha u$ is invertible in $\widehat{R[u,u^{-1}]}$. From this it follows that, if R is not a field, we always have

$$R[u,u^{-1}]^* \neq (\widehat{R[u,u^{-1}]})^* \cap R[u,u^{-1}] \tag{35}$$

and, consequently, by virtue of Proposition 7, $\Sigma \neq S_b$.

Example. Let $R = k[[z]]$, where k is a field. Every $p := \alpha u^k + zq$ with $\alpha \in k^*$ and $q \in R[u,u^{-1}]$ induces an invertible scalar shift operator whose inverse is not continuous if q is not of the type ru^k for some $r \in R$.

Completeness turns out to be crucial in Proposition 7. Indeed, we now show that there are examples of local rings (which are of course not semisimple), for which

Proposition 4 still holds true. Let m be a maximal ideal of $k[z_1,\dots z_s]$ and consider the localization $R := k[z_1,\dots,z_s]_m$. Denote by $\tilde{m}$ the maximal ideal of R. The $\tilde{m}$-completion of R is the ring of formal power series $k[[z_1,\dots,z_s]]$. Consider now a bijective R-morphism

$$\phi: R^{\mathbb{Z}} \to R^{\mathbb{Z}}. \tag{36}$$

It extends to an $\hat{R}$-morphism

$$\hat{\phi}: \hat{R}^{\mathbb{Z}} \to \hat{R}^{\mathbb{Z}} \tag{37}$$

which is still bijective (this can be proven by standard techniques of inverse limits [2]). By virtue of Proposition 7, there exists $p \in k[u,u^{-1}][[z_1,\dots z_s]]$ such that $\hat{\phi} \circ p(\sigma,\sigma^{-1}) = I$. Necessarily, $p(\sigma,\sigma^{-1})R^{\mathbb{Z}} = R^{\mathbb{Z}}$. Write

$$p = \sum_{n_1,\dots,n_s} r_{n_1,\dots,n_s} z_1^{n_1} \cdots z_s^{n_s} \tag{38}$$

with $r_{n_1,\dots n_s} \in k[u,u^{-1}]$. If $p \notin R[u,u^{-1}]$, then the lag of the $r_{n_1,\dots n_s}$'s will not be bounded and thus it would be easy to construct a sequence $x \in k^{\mathbb{Z}}$ such that

$$(p(\sigma,\sigma^{-1})x)(0) = \sum_{n_1,\dots,n_s} (p_{n_1,\dots,n_s}(\sigma,\sigma^{-1})x)(0) z_1^{n_1} \cdots z_s^{n_s} \tag{39}$$

is not a rational function, in particular it is not in $R^{\mathbb{Z}}$. Thus, $p \in R[u,u^{-1}]$ which implies that the inverse of ϕ is itself a scalar shift operator.

The same result can be proven, repeating the argument word by word, for the ring $k\{z_1,\dots,z_n\}$ for $k = \mathbb{R},\mathbb{C}$.

2.4. The PID case: some extra results

In the case of PID's we have some extra results which turn out to be very useful in the applications.

Lemma 8. *Let R be a PID. Consider $p \in R[u,u^{-1}]$ and write $p = ap'$ with $a \in R$ and $p' \in R[u,u^{-1}]$ primitive. Then,*

$$\overline{\operatorname{Im} p(\sigma,\sigma^{-1})} = aR^{\mathbb{Z}}. \tag{40}$$

Proof. Write $p' = \sum_{i=m}^{n} r_i' u^i$. Fix now $s \in \mathbb{N}$ and consider the $s \times (s+n-m)$ matrix

$$P = \begin{bmatrix} r_m' & r_{m+1}' & \cdots & r_n' & 0 & \cdots & 0 \\ 0 & r_m' & \cdots & r_{n-1}' & r_n' & \cdots & 0 \\ & & \ddots & & & & \\ 0 & 0 & \cdots & & & \cdots & r_n' \end{bmatrix}. \tag{41}$$

It is clear that $\operatorname{Im}(P: R^{s+n-m} \to R^s) = (\operatorname{Im} p'(\sigma,\sigma^{-1}))_{|[1,s]}$. Elementary considerations of linear algebra over R show that, since $r_m',\dots,r_n'$ are coprime, P is surjective and so

we have that $(\operatorname{Im} p'(\sigma,\sigma^{-1}))_{|[1,s]} = R^s$. Hence, $\overline{\operatorname{Im} p'(\sigma,\sigma^{-1})} = R^{\mathbb{Z}}$ which yields the result. □

Proposition 9. *Let R be a PID. Consider a polynomial $p \in R[u,u^{-1}]$. Then, $p(\sigma,\sigma^{-1})$ has closed range if and only if $p \in R \cdot \Sigma$*

Proof. One way is trivial, while the other is direct consequence of the previous lemma. □

Corollary 10. *Let R be a PID. Let $m \leq R$ be a maximal ideal and assume that R is complete in the m-adic topology τ. Then,*

$$R[u,u^{-1}] = R \cdot \Sigma. \tag{42}$$

Proof. Let $p \in R[u,u^{-1}]$. Since $R^{\mathbb{Z}}$ is linearly compact and $p(\sigma,\sigma^{-1})$ is τ^{∞}-continuous, it follows that $p(\sigma,\sigma^{-1})$ has a closed range. The result then immediately follows from Proposition 9. □

2.5. Back to difference equations

We now return to difference equations showing how the results established in this section can be used to solve specific problems.

Let $p \in R[u,u^{-1}]$ and factor it as $p = p_1 p_2$ with $p_1 \in \widetilde{S}_b$ and $p_2 \in S_b$. It follows from Proposition 1 that

$$\operatorname{Ker} p(\sigma,\sigma^{-1}) = \operatorname{Ker} p_2(\sigma,\sigma^{-1}). \tag{43}$$

Hence, the difference equation $p(\sigma,\sigma^{-1})x = 0$ has non-trivial solutions if and only if p_2 is not a monomial. This proves that Eqs. (9) and (10) have only the trivial solution, indeed $1 - zu$ and $1 + u + zu^2$ are in $\widetilde{S}_b$ relative to the ring $\mathbb{R}[z][u,u^{-1}]$. On the other hand, $1 + (z-1)u - zu^2 = (1+zu)(1-u)$ and this explains why instead Eq. (11) has non-trivial solutions. As we mentioned in the Introduction, the equation $p_2(\sigma,\sigma^{-1})x = 0$ is easy to solve and the solutions form a free R-module of dimension n if the lag of p_2 is $n+1$. This ends the discussion on the homogeneous problem.

Consider now the non-homogeneous problem

$$p(\sigma,\sigma^{-1})x = y. \tag{44}$$

It can be transformed into the system

$$p_1(\sigma,\sigma^{-1})l = y, \tag{45}$$

$$p_2(\sigma,\sigma^{-1})x = l. \tag{46}$$

As we already mentioned in the Introduction, (46) is easy to solve and there is a useful way to represent its solutions. The difficulty lies in (45). If R is a semisimple ring, it

follows from Proposition 4 that if $p_1 \notin (R[u,u^{-1}])^*$ then, $p_1(\sigma,\sigma^{-1})$ is not surjective. Hence, in this case (44) will not be solvable for all $y \in R^{\mathbb{Z}}$. The difficulty is that in general it is hard to give useful characterization of the image. If we exclude the simple case when $p_1 \in R_0$, Corollary 5 shows that for semisimple rings Im $p_1(\sigma,\sigma^{-1})$ never has finite memory. There is thus no hope to be able to express the image in kernel form. Actually, if R is a semisimple PID, Proposition 9 shows that this image is not even closed. A typical way to try to handle Eq. (45) is the following: enlarge the ring R into a new ring where the scalar shift operator becomes surjective, solve the equation in the new ring, and then check if the solution is in $R^{\mathbb{Z}}$. The easiest way to do this is to work in the field of fractions of R: the drawback of this approach is however that the scalar shift operator becomes surjective but looses injectivity. The non-homogeneous equation in the field of fractions thus give an affine subspace of solutions and it may not be very handy to go and check if one of these is indeed in $R^{\mathbb{Z}}$. A technique which turns out to be useful in certain cases is to enlarge the ring by taking the completion with respect to some suitable m-adic topology. We now present a series of examples illustrating this technique.

Example 1. Let $R = \mathbb{R}[z]$ and let $p = 1 - zu \in R[u]$. $p \in \widetilde{S}_b$, hence $p(\sigma,\sigma^{-1}) = 1 - z\sigma$ is not surjective and actually, by Proposition 8 the image is dense in $R^{\mathbb{Z}}$. Consider the (z)-completion $\mathbb{R}[[z]]$. It follows from Proposition 7 and the Example following it, that

$$p(\sigma,\sigma^{-1}): \hat{R}^{\mathbb{Z}} \to \hat{R}^{\mathbb{Z}} \tag{47}$$

is bijective and the inverse is given by

$$q(\sigma,\sigma^{-1}) = \sum_{i=0}^{+\infty} z^i \sigma^i. \tag{48}$$

The solution of

$$p(\sigma,\sigma^{-1})x = y \tag{49}$$

in $\hat{R}^{\mathbb{Z}}$ is thus given by

$$x(t) := \sum_{i=0}^{+\infty} z^i y(t+i). \tag{50}$$

This gives a way also to study (49) in $R^{\mathbb{Z}}$: given $y \in R^{\mathbb{Z}}$, there exists $x \in R^{\mathbb{Z}}$ which solves (49) if and only if

$$\sum_{i=0}^{+\infty} z^i y(t+i) \in \mathbb{R}[z] \quad \forall t \in \mathbb{Z} \tag{51}$$

and if this happens, then the solution is given by (50). Let us work a bit on condition (51). It is actually sufficient that

$$\sum_{i=0}^{+\infty} z^i y(i) \in \mathbb{R}[z]. \tag{52}$$

Writing $y(i) = \sum_j y_{i,j} z^j$, we thus find the equivalent condition

$$\exists n_0 \in \mathbb{N} \quad \text{such that} \quad \sum_{\substack{i+j=n \\ i \geq 0}} y_{i,j} = 0 \quad \forall n \geq n_0. \tag{53}$$

Notice in particular that, as it was easy to understand from the structure of p itself, for all $y \in R^{\mathbb{Z}}$ which have right finite support, (49) admits solution in $R^{\mathbb{Z}}$. Moreover, if $y \in \mathbb{R}^{\mathbb{Z}}$, then, it follows from (53) that (49) admits solution in $R^{\mathbb{Z}}$ if and only if y has right finite support. It is easy to check that this is no longer true if y depends on z.

Example 2. Let $R = \mathbb{R}[z]$ and let $p = 1 + z - zu \in R[u]$. As in the above example $p(\sigma, \sigma^{-1})$ has dense, not closed image. Passing to the completion and repeating the same arguments as above (on the completion again the scalar shift operator becomes bijective), then we get that the non-homogeneous equation associated with p and $y \in R^{\mathbb{Z}}$ has a solution if and only if

$$\forall t \in \mathbb{Z} \ \exists n_0 \in \mathbb{N} \quad \text{such that} \quad \sum_{i+k=n} \sum_{j=0}^{k} \binom{k}{j} y_{t+j,i} = 0 \quad \forall n \geq n_0 \tag{54}$$

and if this happens the unique solution is given by

$$x(t) := \sum_{k=0}^{+\infty} \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} z^k y(t+j). \tag{55}$$

The next example shows how things can also go bad.

Example 3. Let $R = \mathbb{R}[z]$ and let $p = 1 + u + zu^2 \in R[u]$. Again $p(\sigma, \sigma^{-1})$ has dense not closed image. What happens in the completion $\hat{R} = \mathbb{R}[[z]]$? Since p is primitive as a polynomial in $\hat{R}[u, u^{-1}]$, it follows from Corollary 10 that $p(\sigma, \sigma^{-1})$ is surjective as an operator on $\hat{R}^{\mathbb{Z}}$; on the other hand, it does not admit an inverse in the ring $\mathbb{R}[u, u^{-1}][[z]]$, consequently it cannot be injective. This makes this example more difficult than previous ones: we will not get linear conditions as above for the solvability of the non-homogeneous problem in $R^{\mathbb{Z}}$. However, let us make some other considerations. Note first that p is no longer irreducible on $\hat{R}[u, u^{-1}]$. Indeed, let $\alpha \in \mathbb{R}[[z]]$ such that $(1 + z\alpha)^2 = 1 - 4z$. Then,

$$\begin{aligned} p &= z \left(u - \frac{-1 - \sqrt{1 - 4z}}{2z} \right) \left(u - \frac{-1 + \sqrt{1 - 4z}}{2z} \right) \\ &= \left(zu + \frac{2 + z\alpha}{2} \right) \left(u - \frac{\alpha}{2} \right). \end{aligned} \tag{56}$$

Note that

$$p_1 = zu + \frac{2+z\alpha}{2} = 1 + z\left(u + \frac{\alpha}{2}\right) \tag{57}$$

induces an invertible scalar shift operator. On the other hand, since $\alpha(0) \neq 0$, we have that $u - \alpha/2$ is bimonic in $\hat{R}[u, u^{-1}]$ so that the corresponding scalar shift operator is surjective but not injective. The inverse of $p_1(\sigma, \sigma^{-1})$ in $\mathbb{R}[[z]]^{\mathbb{Z}}$ is given by

$$p_1(\sigma, \sigma^{-1})^{-1} = \sum_{k=0}^{+\infty} z^k \left(\sigma + \frac{\alpha}{2}\right)^k, \tag{58}$$

so that the set of all the solutions of $p(\sigma, \sigma^{-1})x = y$ in $\mathbb{R}[[z]]^{\mathbb{Z}}$ is given by

$$x_\lambda(t) = \left(\frac{\alpha}{2}\right)^t \lambda + \sum_{j=0}^{t-1} \left(\frac{\alpha}{2}\right)^j \left(\sum_{k=0}^{+\infty} z^k \left(\sigma + \frac{\alpha}{2}\right)^k y\right)(t-1-j) \tag{59}$$

as λ varies in $\mathbb{R}[[z]]$. This formula is not very appealing and it may be practically impossible to establish if, given $y \in \mathbb{R}[z]^{\mathbb{Z}}$, there exists $\lambda \in \mathbb{R}[[z]]$ such that (59) is a polynomial for all $t \in \mathbb{Z}$. However, it may be possible to establish if there is at least a solution which is analytic in a neighborhood of 0. Notice first that $\alpha \in \mathbb{R}\{z\}$ so that the decomposition $p = p_1 p_2$ still holds in the smaller ring $\mathbb{R}\{z\}[u, u^{-1}]$. It is therefore clear that such a solution exists if and only if

$$\sum_{k=0}^{+\infty} z^k \left(\sigma + \frac{\alpha}{2}\right)^k y \in \mathbb{R}\{z\}^{\mathbb{Z}} \tag{60}$$

and if this is true then, all the possible analytic solutions are given by (59) as λ varies in $\mathbb{R}\{z\}$. While it remains impossible to find necessary and sufficient conditions for (60) to hold true, it is nevertheless possible to find interesting sufficient conditions. Fix first $\delta > 0$ such that α is convergent and bounded by 1 in the ball $B(0, \delta)$. Consider then a $y \in \mathbb{R}^{\mathbb{Z}}$. We have

$$\left|\left(\left(\sigma + \frac{\alpha}{2}\right)^k y\right)(t)\right| = \left|\sum_{j=0}^{k} \binom{k}{j} \left(\frac{\alpha}{2}\right)^j y(t+k-j)\right| \tag{61}$$

$$\leq \sum_{j=0}^{k} \binom{k}{j} \left(\frac{1}{2}\right)^j \sup_{t \leq i \leq t+k} |y(i)| = \sup_{t \leq i \leq t+k} |y(i)| \left(\frac{3}{2}\right)^k. \tag{62}$$

From this it follows that (60) holds if $y \in \mathbb{R}^{\mathbb{Z}}$ has exponential growth for positive times. It also follows that (60) holds if $y \in \mathbb{R}[z]^{\mathbb{Z}}$ has bounded degree and the coefficients have exponential growth for positive times.

The situation of the previous example regarding the local analytic case contains general facts which are worth discussing a bit longer. Let k be the real or complex field and let $p \in k\{z_1, \ldots z_n\}[u, u^{-1}]$. Assume that $p \in \widetilde{S}_b$. A priori p might have

bimonic factors in $k[[z_1,\dots z_n]][u,u^{-1}]$. However, it follows from an easy application of an important result by Artin [1] that this is not the case. p is in $\widetilde{S}_b$ also relative to the ring $k[[z_1,\dots z_n]][u,u^{-1}]$. Therefore, $p(\sigma,\sigma^{-1})$ remains injective on $k[[z_1,\dots z_n]]^{\mathbb{Z}}$. This proves the following interesting result.

Proposition 11. *Let* $R = k\{z_1,\dots z_s\}$ *and let* $\hat{R} = k[[z_1,\dots z_s]]$. *Consider* $p \in k\{z_1,\dots z_s\}[u,u^{-1}]$. *Then,*

$$\overline{\operatorname{Ker} p(\sigma,\sigma^{-1})_{|R^{\mathbb{Z}}}} = \operatorname{Ker} p(\sigma,\sigma^{-1})_{|\hat{R}^{\mathbb{Z}}}, \tag{63}$$

where the closure is with respect to the product topology τ^{∞}.

Going back to the non-homogeneous problem, we thus have that injectivity is in this case never lost when we pass to the completion. In the general case it is however not easy to solve the non-homogeneous problem even on $k[[z_1,\dots z_s]]^{\mathbb{Z}}$. The case $s = 1$ however, because of Proposition 7 and Corollary 10, is completely solved and this gives a good way also to study the problem on $k\{z\}^{\mathbb{Z}}$ as it was done in Example 3. In particular it can be easily shown that we can always find exponential growth type sufficient conditions.

We now come to other examples.

Example 4. Let $R = \mathbb{Z}$ and let $p = 1 - 2u \in \mathbb{Z}[u]$. This example resembles Example 1 and indeed it will be handled in a similar way though certain considerable differences will appear in the end. Again $p(\sigma,\sigma^{-1}) = 1 - 2\sigma$ has dense, not closed image in $R^{\mathbb{Z}}$. Denote by $\hat{R}$ the ring of 2-adic numbers namely, the completion of $\mathbb{Z}$ with respect to the maximal ideal (2): elements of $\hat{R}$ can be represented as power series $\sum a_i 2^i$ where $a_i \in \{0,1\}$. $p(\sigma,\sigma^{-1})$ is bijective on $\hat{R}^{\mathbb{Z}}$ and the inverse is given by

$$q(\sigma,\sigma^{-1}) = \sum_{i=0}^{+\infty} 2^i \sigma^i. \tag{64}$$

Hence, given $y \in \mathbb{Z}^{\mathbb{Z}}$, there exists $x \in \mathbb{Z}^{\mathbb{Z}}$ such that

$$p(\sigma,\sigma^{-1})x = y \tag{65}$$

if and only if

$$x(t) := \sum_{i=0}^{+\infty} 2^i y(t+i) \in \mathbb{Z} \quad \forall t \in \mathbb{Z}. \tag{66}$$

Apart from Example 1 we cannot find any simple condition similar to (53): the difficulty lies in the fact that integers inside the 2-adic complement do not simply correspond to 'polynomials in 2' (for instance we have that $-1 = \sum 2^i$).

Example 5. Let $R = \mathbb{R}[z_1,z_2]$ and consider $p = z_1 - z_2 u$. $p(\sigma,\sigma^{-1})$ is injective. We cannot say very much about the image except that it does not have finite memory by

Corollary 5. In the completion $\mathbb{R}[[z_1, z_2]]$, $p(\sigma, \sigma^{-1})$ is still injective but, however, not surjective: otherwise p would be invertible in $\mathbb{R}[u, u^{-1}][[z_1, z_2]]$ while it is not. However, it becomes bijective if we invert z_1, namely if we work in the ring $\mathbb{R}[[z_1, z_2]][z_1^{-1}]$. The inverse is given by

$$q(\sigma, \sigma^{-1}) := z_1^{-1} \sum_{k=0}^{+\infty} (-1)^k z_1^{-k} z_2^k \sigma^k. \tag{67}$$

By repeating the arguments of Example 1, we can easily prove that given a $y \in R^{\mathbb{Z}}$, there exists $x \in R^{\mathbb{Z}}$ such that $p(\sigma, \sigma^{-1})x = y$ if and only if

$$\forall t \in \mathbb{Z} \quad \sum_{l+j=n} y_{t+j,l}(z_1) z_1^{-j} \in \mathbb{R}[z_1] \quad \forall n \in \mathbb{N}, \tag{68}$$

or equivalently

$$\forall t \in \mathbb{Z} \ \exists n_0 \in \mathbb{N} \quad \sum_{l+j=n} y_{t+j,l}(z_1) z_1^{-j} = 0 \quad \forall n \geq n_0, \tag{69}$$

where $y(t) = \sum_l y_{t,l}(z_1) z_2^l$ with $y_{t,l} \in \mathbb{R}[z_1]$.

3. Matrix shift operators

3.1. Polynomial matrices

Let A be a factorial domain. The rank of $M \in A^{l \times q}$ is defined as the usual rank with respect to the field of fractions of A. $M \in A^{l \times q}$ is said to be full column (resp. row) rank if its rank is equal to q (resp. l).

Let $S \subseteq A$ be a multiplicatively closed subset. $M \in A^{l \times q}$ is said to be *right S-factor prime* if it has full column rank and if $M = M'F$ with $M' \in A^{l \times q}$ and $F \in A^{q \times q}$ yields $\det F \in S$. A matrix M is said to be *left S-factor prime*, if M^{T} is right S-factor prime.

The following was essentially proved in [7, 9]. In the present form it can be found in [5].

Theorem 12. *Let R be a UFD. Let $S \subseteq R[u, u^{-1}]$ be a saturated multiplicatively closed set. Let $M \in R[u, u^{-1}]^{l \times q}$ be full column rank. Consider the following facts:*

(1) *M is right S-factor prime.*

(2) *Every common factor of $q \times q$ minors of M is in S.*

(3) *There exists $N \in R[u, u^{-1}]^{q \times l}$ and $a \in S$ such that*

$$NM = aI. \tag{70}$$

Then, $(1) \Leftarrow (2) \Leftarrow (3)$ If $S^{-1}R[u, u^{-1}]$ is a PID, then $(1) \Leftarrow (2) \Leftrightarrow (3)$. If, moreover, R is a PID, then all the previous conditions are equivalent.

The following is in [9].

Lemma 13. *Let R be a PID. Let $A \in R[u,u^{-1}]^{l\times l}$ with $\det A = f_1 f_2$ where $f_1, f_2 \in R[u,u^{-1}] \setminus \{0\}$. Then, there exist $F_i \in R[u,u^{-1}]^{l\times l}$ with $\det F_i = f_i$ for $i = 1,2$ such that*

$$A = F_1 F_2. \tag{71}$$

We have another useful factorization result.

Lemma 14. *Let R be a PID. Let $M \in R[u,u^{-1}]^{l\times q}$ be a matrix of rank r. Then, there exist matrices $M_1 \in R[u,u^{-1}]^{l\times r}$ and $M_2 \in R[u,u^{-1}]^{r\times q}$ such that $M = M_1 M_2$. Moreover, if $S \subseteq R[u,u^{-1}]$ is any saturated multiplicatively closed subset, then M_1 can be chosen to be right S-factor prime and M_2 left $\widetilde{S}$-factor prime.*

Proof. With standard techniques on the ring $F[u,u^{-1}]$, where F is the field of fractions of R, we obtain matrices $N_1 \in R[u,u^{-1}]^{l\times r}$ and $N_2 \in R[u,u^{-1}]^{r\times q}$, and $a \in R_0$ such that $aM = N_1 N_2$. Factor now $N_2 = N_2' M_2$ with $N_2' \in R[u,u^{-1}]^{r\times r}$ and $\det N_2' \in R_0$, and with $M_2 \in R[u,u^{-1}]^{r\times q}$ left $\widetilde{R}_0$-factor prime. It is now easy to show that there exists $M_1 \in R[u,u^{-1}]^{l\times r}$ such that $M = M_1 M_2$ (see [9]). The last part is now a consequence of Lemma 13. □

3.2. Matrix shift operators

Theorem 15. *Let R be a UFD. Let $M \in R[u,u^{-1}]^{l\times q}$. Consider the following facts:*

(1) *$M(\sigma,\sigma^{-1})$ is injective.*

(2) *There exist $X \in R[u,u^{-1}]^{q\times l}$ and $a \in \widetilde{S}_b$ such that*

$$XM = aI. \tag{72}$$

(3) *M is right $\widetilde{S}_b$-factor prime.*

Then, (1)$\Leftrightarrow$(2)$\Rightarrow$(3). Moreover, if R is a PID we also have (2)$\Leftarrow$(3).

Proof. (2)$\Rightarrow$(3) is trivial and (2)$\Rightarrow$(1) is a consequence of Proposition 1.

(1)$\Rightarrow$(2) M is full column rank because otherwise it would be possible to find a non-zero sequence with finite support in the kernel of $M(\sigma,\sigma^{-1})$. By standard diagonalization techniques, there exist $U \in R[u,u^{-1}]^{l\times l}$, $V, \Lambda \in R[u,u^{-1}]^{q\times q}$ with $\det U, \det V \in R_0$ and Λ diagonal, and $r \in R_0$ such that

$$rM = U \begin{bmatrix} \Lambda \\ 0 \end{bmatrix} V. \tag{73}$$

Clearly, $\Lambda(\sigma,\sigma^{-1})$ is injective and this yields $\det \Lambda \in \widetilde{S}_b$ by Proposition 1. Write

$$\mathrm{Adj}(U) = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{74}$$

with U_1 in $R[u,u^{-1}]^{q\times l}$. We thus obtain from (73)

$$rU_1M = (\det U)\Lambda V. \tag{75}$$

Hence, if $Z = \mathrm{Adj}(\Lambda V)$, we have

$$rZU_1M = (\det U \det \Lambda \det V)I. \tag{76}$$

Hence (2) is proven with $X = rZU_1$ and $a = \det U \det \Lambda \det V$.

If R is a PID, (2)$\Leftarrow$(3) follows from Theorem 12. □

For the rest of this subsection we will assume that R is a PID. Before analyzing the class of surjective matrix shift operators, we need some preliminary definitions and results. Consider $R_b := S_b^{-1}R[u,u^{-1}]$. It follows from the arguments in [8] (Chapter IV), that if R is a PID, also R_b is a PID. For the sake of completeness we here give a completely elementary proof of this fact. Let $\mathscr{I}$ be any ideal in $R[u,u^{-1}]$ such that $\mathscr{I}\cap R \neq \{0\}$. For all $n \in \mathbb{N}$, define the following ideals of R

$$J_n := \{a \in R: \exists p \in R[u] \ \deg p < n \text{ with } au^n + p \in \mathscr{I}\}. \tag{77}$$

Let $a_n \in R$ be such that $J_n = (a_n)$. Clearly, $a_n | a_{n-1}$. Let $q_n \in \mathscr{I}$ be such that

$$q_n = a_n u^n + \tilde{q}_n, \tag{78}$$

where $\tilde{q}_n \in R[u]$ and $\deg \tilde{q}_n < n$.

Lemma 16. *Let $\mathscr{I}$ be any ideal in $R[u,u^{-1}]$ such that $\mathscr{I}\cap R \neq \{0\}$. Let $a_n \in R$ and $q_n \in R[u]$ be defined as above. Then $a_n | q_n$ for all n.*

Proof. The result is obvious for $n = 0$. Assume it is true for $n \leq k-1$ and prove it for $n = k$. By contradiction assume that $a_k \not| q_k$ and let $m < k$ such that there exist $f, g \in R[u]$ such that

$$q_k = a_k f + g \tag{79}$$

and such that $\deg g = m$ and a_k does not divide the leading coefficient c of g. Consider now $q_m = a_m(u^m + \bar{q}_m)$, where $\bar{q}_m \in R[u]$ with $\deg \bar{q}_m < m$. We know that $a_m = ba_k$ for some $b \in R \setminus \{0\}$. Let $d, e \in R[u]$ with $\deg e < m$ such that

$$e := f - d(u^m + \bar{q}_m). \tag{80}$$

Consider

$$\begin{aligned} bq_k - dq_m &= ba_k f + bg - da_m(u^m + \bar{q}_m) \\ &= a_m(f - d(u^m + \bar{q}_m)) + bg \\ &= a_m e + bg \in \mathscr{I}. \end{aligned} \tag{81}$$

Notice that the leading coefficient of $bq_k - dq_m$ equals the leading coefficient bc of bg and that $\deg g = m$. Hence

$$bc = aa_m = aba_k \tag{82}$$

and so $c = aa_k$ which is a contradiction. □

Proposition 17. *If R is a PID, then the ring $R_b := S_b^{-1}R[u,u^{-1}]$ is also a PID.*

Proof. Let I be an ideal of R_b and let $e_1,\dots,e_k \in R[u,u^{-1}]$ be a set of generators for I (as an ideal of R_b). Let $e \in R[u,u^{-1}]$ be the greatest common divisor (in $R[u,u^{-1}]$) of $e_1,\dots,e_k$. Set $f_i = e^{-1}e_i$. Denote by $\mathscr{I}$ the ideal in $R[u,u^{-1}]$ generated by $f_1,\dots,f_k$. It is now sufficient to prove that $\mathscr{I} \cap S_b \neq \emptyset$. Notice that, by construction, $\mathscr{I} \cap R \neq \{0\}$. Define the sequences a_n and q_n as in Lemma 16 and let $a \in R$ be the greatest common divisor of the a_n in R. It is clear that we can express any $p \in \mathscr{I}$ as follows:

$$p = u^{-l}\sum_{n=0}^{s} \alpha_n q_n, \tag{83}$$

where $\alpha_n \in R$ and l is a suitable non-negative integer. Since a divides each q_n, then it divides also p. We can argue that a divides $f_1,\dots,f_k$ and since they are coprime, then a must be an invertible element in R. Hence a_n is invertible for some n and, consequently, the corresponding q_n is a monic polynomial. In a similar way we can show that there exists a polynomial in $\mathscr{I} \cap R[u^{-1}]$ whose lowest degree term has invertible coefficient. This easily implies the existence of a bimonic polynomial in $\mathscr{I}$. This completes the proof. □

We are now in position to give the characterization of the class of surjective matrix shift operators.

Theorem 18. *Let R be a PID and let $M \in R[u,u^{-1}]^{l\times q}$. Then the following are equivalent:*

(1) *$M(\sigma,\sigma^{-1})$ is surjective.*

(2) *There exists $X \in R[u,u^{-1}]^{q\times l}$ such that*

$$MX = aI, \tag{84}$$

where $a \in \Sigma$.

(3) *M is left Σ-factor prime.*

Proof. Since $S_b \subseteq \Sigma$, it follows from Proposition 17 that $\Sigma^{-1}R[u,u^{-1}]$ is a PID. Therefore, (3) ⇒ (2) follows from Theorem 12 applied to M^{T}.

(2) ⇒ (1) is trivial.

(1) ⇒ (3) It is immediate to check that M must be full row rank. Suppose $M = \Lambda M'$ with $\Lambda \in R[u,u^{-1}]^{l\times l}$ and $M' \in R[u,u^{-1}]^{l\times q}$. Passing through the Smith form of Λ in the PID $\Sigma^{-1}R[u,u^{-1}]$, we find $U,V \in R[u,u^{-1}]^{l\times l}$ with $\det U, \det V \in \Sigma$ such that

$D := U\Lambda V$ is a diagonal matrix. Clearly, $D(\sigma,\sigma^{-1})$ is a surjective shift operator and, therefore, $\det D \in \Sigma$. This yields $\det \Lambda \in \Sigma$ and this completes the proof. □

Theorem 19. *Let R be a PID and let $M \in R[u,u^{-1}]^{l\times q}$. Then the following facts are equivalent*:

(1) *$M(\sigma,\sigma^{-1})$ is an open surjection.*

(2) *There exists $X \in R[u,u^{-1}]^{q\times l}$ such that*

$$MX = aI, \tag{85}$$

where $a \in S_b$.

(3) *M is left S_b-factor prime.*

Proof. Since $S_b^{-1}R[u,u^{-1}]$ is a PID, (3)⇒(2) follows from Theorem 12 applied to M^{T}.

(2)⇒(1) Let $w_n \in (R^l)^{\mathbb{Z}}$ converging to zero. Then, there exists, $v_n \in (R^l)^{\mathbb{Z}}$ converging to zero such that $a(\sigma,\sigma^{-1})v_n = w_n$. Put $u_n := X(\sigma,\sigma^{-1})v_n$. Then $u_n \to 0$ and $M(\sigma,\sigma^{-1})u_n = w_n$. This yields (1).

(1)⇒(3) Write $M = M_1M_2$ with M_1 right $\widetilde{S}_b$-factor prime and M_2 left S_b-factor prime. We have that $M_1(\sigma,\sigma^{-1})$ is open. Using the same arguments of the proof of Proposition 2, we obtain that M_1 must be square and invertible. □

Finally we want to study closed range shift operators. We first need a lemma.

Lemma 20. *Let R be a PID and let $M \in R[u,u^{-1}]^{l\times q}$ be such that $M(\sigma,\sigma^{-1})$ has closed range. Let $K \subseteq (R^q)^{\mathbb{Z}}$ be a closed R-submodule such that*

$$(aR^q)^{\mathbb{Z}} \subseteq K \subseteq (R^q)^{\mathbb{Z}} \tag{86}$$

for some $a \in R_0$. Then $M(\sigma,\sigma^{-1})(K)$ is closed.

Proof. $M(\sigma,\sigma^{-1})$ induces a surjection

$$\phi\colon ((R/aR)^q)^{\mathbb{Z}} \to \operatorname{Im} M(\sigma,\sigma^{-1})/a(\operatorname{Im} M(\sigma,\sigma^{-1})). \tag{87}$$

Clearly, $K/(aR^q)^{\mathbb{Z}}$ is closed inside $((R/aR)^q)^{\mathbb{Z}}$, and since this last one is linearly compact [15], we have that $M(\sigma,\sigma^{-1})(K)/a(\operatorname{Im} M(\sigma,\sigma^{-1})) = \phi(K/(aR^q)^{\mathbb{Z}})$ is closed inside $\operatorname{Im} M(\sigma,\sigma^{-1})/a(\operatorname{Im} M(\sigma,\sigma^{-1}))$. This implies that $M(\sigma,\sigma^{-1})(K)$ is closed in $(R^l)^{\mathbb{Z}}$. □

We are now ready to characterize closed range shift operators. First, we introduce another notion of primeness which is a generalization of previous ones. Let $S \subseteq R_0$ be a multiplicatively closed set and let $M \in R[u,u^{-1}]^{l\times q}$ be a matrix of rank r. M is said to be S-factor prime if given any factorization of type $M = M_1\Lambda M_2$, where $M_1 \in R[u,u^{-1}]^{l\times r}$, $\Lambda \in R[u,u^{-1}]^{r\times r}$, and $M_2 \in R[u,u^{-1}]^{r\times q}$, we have that $\det \Lambda \in S$. It is easy to see that if $r = l$ (resp. $r = q$), then M is S-factor prime if and only if it is left (resp. right) S-factor prime.

Theorem 21. *Let R be a PID and let M be a rank $r > 0$ matrix in $R[u,u^{-1}]^{l\times q}$. Then the following are equivalent:*

(1) *$M(\sigma,\sigma^{-1})$ has a closed range.*

(2) *There exist matrices $X \in R[u,u^{-1}]^{r\times l}$, $Y \in R[u,u^{-1}]^{q\times r}$ such that*

$$XMY = aI, \tag{88}$$

where $a \in R_0\Sigma$.

(3) *M is $R_0\Sigma$-factor prime.*

Proof. (1)$\Rightarrow$(3) Passing through the Smith form of M in the PID $F[u,u^{-1}]$, where F is the field of fractions of R, we obtain that there exist matrices $U \in R[u,u^{-1}]^{l\times l}$, $V \in R[u,u^{-1}]^{q\times q}$ with $a = \det U, b = \det V \in R_0$ and a diagonal non-singular $D \in R[u,u^{-1}]^{r\times r}$ such that

$$UMV = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}. \tag{89}$$

Notice that $\operatorname{Im} V(\sigma,\sigma^{-1})$ is closed and

$$b(R^q)^{\mathbb{Z}} \subseteq \operatorname{Im} V(\sigma,\sigma^{-1}) \subseteq (R^q)^{\mathbb{Z}}. \tag{90}$$

Hence by Lemma 20, $\operatorname{Im}(MV)(\sigma,\sigma^{-1})$ is also closed. Since $U(\sigma,\sigma^{-1})$ is invertible on its image, it follows that $\operatorname{Im} D(\sigma,\sigma^{-1})$ is closed. By Proposition 9, this implies that $\det D \in R_0\Sigma$. In particular, we have that UMV is $R_0\Sigma$ factor prime and this immediately yields that M is also $R_0\Sigma$ factor prime.

(3)$\Rightarrow$(2) Write $M = M_1M_2$ as in Lemma 14. It is clear that M_1 is right $R_0\Sigma$-factor prime and M_2 is left $R_0\Sigma$-factor prime. The result then follows from Theorem 12.

(2)$\Rightarrow$(1) Consider the factorization $M = M_1M_2$ as in Lemma 14. Since we have $XM_1M_2Y = aI$, it follows that $\det(XM_1), \det(M_2Y) \in R_0\Sigma$. It easily follows from Theorem 12 that by a suitable choice of the factorization we can assume that M_1 is right R_0-factor prime and M_2 is left Σ-factor prime. By Theorem 18, $\operatorname{Im} M(\sigma,\sigma^{-1}) = \operatorname{Im} M_1(\sigma,\sigma^{-1})$. Since, by Theorem 12, $M_1(\sigma,\sigma^{-1})$ admits a continuous inverse on its image, the result now follows. □

3.3. Systems of difference equations

Consider first the homogeneous equation

$$M(\sigma,\sigma^{-1})v = 0, \tag{91}$$

where $M \in R^{l\times q}[u,u^{-1}]$. We would like to note first of all that Theorem 15 furnishes a way to establish for any Noetherian factorial domain if, given a matrix M, $\operatorname{Ker} M(\sigma,\sigma^{-1})$ is empty or not: indeed condition (1) of Theorem 15 is equivalent, by Theorem 12, to the more concrete condition that the common factors of the principal

minors of the matrix M must be in $\widetilde{S}_b$. If R is a PID it is possible to say much more about $\operatorname{Ker} M(\sigma, \sigma^{-1})$.

Assume that M has rank r. Using Lemma 14 we can factor $M = M_1 M_2$ where $M_1 \in R^{l\times r}[u, u^{-1}]$ is right $\widetilde{S}_b$-factor prime and $M_2 \in R^{r\times q}[u, u^{-1}]$ is left S_b-factor prime. By virtue of Theorem 15, we have that

$$\operatorname{Ker} M(\sigma, \sigma^{-1}) = \operatorname{Ker} M_2(\sigma, \sigma^{-1}). \tag{92}$$

In studying (91) we can therefore assume that M is left S_b-factor prime. Under this condition it can be determined a characterization of the class of kernels that are controllable and so that admit an image representation.

Theorem 22. *Let R be a PID and let $M \in R^{l\times q}[u, u^{-1}]$ be left S_b-factor prime. Then, the following are equivalent:*

(1) $\mathscr{B} := \operatorname{Ker} M(\sigma, \sigma^{-1})$ *is controllable.*

(2) *There exists $X \in R^{q\times l}[u, u^{-1}]$ such that*

$$MX = I. \tag{93}$$

(3) *The ideal generated by the $l \times l$ minors of M coincides with $R[u, u^{-1}]$.*

Proof. (2) $\Rightarrow$ (1) It is easy to verify that

$$\operatorname{Ker} M(\sigma, \sigma^{-1}) = \operatorname{Im}(I - XM)(\sigma, \sigma^{-1})$$

and this implies that $\mathscr{B}$ is controllable.

(3) $\Rightarrow$ (2) Let $m_1, m_2, \ldots, m_s$ be the $l \times l$ minors of M. There exist $h_1, h_2, \ldots, h_s \in R[u, u^{-1}]$ such that

$$\sum_{i=1}^{s} h_i m_i = 1. \tag{94}$$

Suppose that S_i is the selection matrix (i.e. a matrix in $R^{q\times l}$ with only zeros and ones), such that $m_i = \det(MS_i)$. Then

$$I = \sum_{i=1}^{s} h_i m_i I = \sum_{i=1}^{s} h_i (\det MS_i) I = M \left(\sum_{i=1}^{s} h_i S_i \operatorname{Adj}(MS_i) \right) \tag{95}$$

(1) $\Rightarrow$ (3) First we want to show that M is left $(R[u, u^{-1}])^*$-factor prime. Factor $M = FN$ with $F \in R^{l\times l}[u, u^{-1}]$ with $\det F \in S_b$ and $N \in R^{l\times q}[u, u^{-1}]$ left $(R[u, u^{-1}])^*$-factor prime. Since $\mathscr{B}$ is controllable, then there exists a polynomial matrix P such that $\mathscr{B} = \operatorname{Im} P(\sigma, \sigma^{-1})$. Notice that $MP = 0$ and so also $NP = 0$ holds true. From these facts it is easy to verify that

$$\operatorname{Ker} N(\sigma, \sigma^{-1}) = \operatorname{Ker} M(\sigma, \sigma^{-1}). \tag{96}$$

Let $X \in R^{q\times l}[u,u^{-1}]$ be such that $NX = aI$ for some $a \in R_0$. Then, if $w \in \operatorname{Ker} F(\sigma,\sigma^{-1})$ we have that

$$0 = aF(\sigma,\sigma^{-1})w = (FNX)(\sigma,\sigma^{-1})w = (MX)(\sigma,\sigma^{-1})w.$$

By (96), we also have

$$aw = (NX)(\sigma,\sigma^{-1})w = 0$$

that implies that $w = 0$. This means that $F(\sigma,\sigma^{-1})$ is injective. By Theorem 15, F is invertible and so M is left $(R[u,u^{-1}])^*$-factor prime.

Let m be a maximal ideal in R. Consider the residue field $k := R/m$. Let $\bar{M}$ be the quotient projection of M over $k^{l\times q}[u,u^{-1}]$ and let $\bar{\mathscr{B}} := \operatorname{Ker}\bar{M}(\sigma,\sigma^{-1}) \subseteq (k^q)^{\mathbb{Z}}$. It can easily be proved that since $M(\sigma,\sigma^{-1})$ is surjective, then $\bar{M}(\sigma,\sigma^{-1})$ is surjective. We want to show that $\bar{\mathscr{B}}$ is controllable. This follows easily from the fact that any $\bar{w} \in \bar{\mathscr{B}}$ admits a representative in $\mathscr{B}$. Actually, if $\bar{w} \in \operatorname{Ker}\bar{M}(\sigma,\sigma^{-1})$ and w is a representative of $\bar{w}$, then $M(\sigma,\sigma^{-1})w = kv$ and, since $M(\sigma,\sigma^{-1})$ is surjective, there exists u such that $v = M(\sigma,\sigma^{-1})u$. This implies that $M(\sigma,\sigma^{-1})(w - ku) = 0$ and so $w - ku$ is a representative of $\bar{w}$ that is in $\mathscr{B}$.

Let J be the ideal generated by the $l \times l$ minors of M. Since M is left $(R[u,u^{-1}])^*$-factor prime, then there exists $a \in R_0$ such that $a \in J$. Let k be a prime in R such that $a = a'k$. From the previous considerations we can argue that the projection of J on $R/(k)[u,u^{-1}]$, that coincides with the ideal generated by the $l \times l$ minors of $\bar{M}$, cover all $R/(k)[u,u^{-1}]$, and so there exists $p \in R[u,u^{-1}]$ such that $1 + kp \in J$. This implies that

$$a' = a'(1 + kp) - ap \in J.$$

A simple induction argument then shows that $1 \in J$. □

Remark. In applying Theorem 22 it is necessary to be able to determine if an ideal generated by a finite family of polynomials contains 1. Note first that, given an ideal I in $R[u,u^{-1}]$ generated by $g_1,\ldots,g_n$, then $I \cap R \neq \{0\}$ if and only if $g_1,\ldots,g_n$ are coprime as polynomials in $F[u,u^{-1}]$, where F is the field of fractions of R. Therefore the condition $I \cap R \neq \{0\}$ can be verified by using the Euclidean algorithm in the Euclidean domain $F[u,u^{-1}]$.

Suppose now that we have found that $a \in R_0$ is in I and let

$$M = \{m \in \max(R) : a \in m\}.$$

Again by Euclidean algorithm in $R/m[u,u^{-1}]$ it is possible to verify if the ideal in $R/m[u,u^{-1}]$ generated by $g_1 + m[u,u^{-1}],\ldots,g_n + m[u,u^{-1}]$ coincides with $R/m[u,u^{-1}]$. This is the case if and only if $I = R[u,u^{-1}]$. It is easy to see that since R is a principal

ideal domain, the set M is finite. Therefore the check of the equality $I = R[u, u^{-1}]$ requires a finite number of applications of the Euclidean algorithm.

Note that the previous theorem provides a technique for obtaining an image representation of $\mathscr{B}$ ($\mathscr{B} = \operatorname{Im}(I - XM)(\sigma, \sigma^{-1})$). This image representation however provides a non-injective parametrization. This drawback can be overcome in the following way. The fact that there exists X such that $MX = I$ implies that there exists $M' \in R^{(q-l)\times q}[u, u^{-1}]$ be such that

$$\begin{bmatrix} M \\ M' \end{bmatrix} \tag{97}$$

is invertible. Let $X \in R^{q\times l}[u, u^{-1}]$, $X' \in R^{q\times(q-l)}[u, u^{-1}]$ be such that $[X\ X']$ is its inverse. It is easy to verify that

$$\mathscr{B} = \operatorname{Im} X'(\sigma, \sigma^{-1}) \tag{98}$$

and that $X'(\sigma, \sigma^{-1})$ is injective.

As we have seen, when $\mathscr{B} = \operatorname{Ker} M(\sigma, \sigma^{-1})$ is controllable, then the parametrization of the solutions of the homogeneous difference equation (91) is particularly simple.

In general, given $\mathscr{B} = \operatorname{Ker} M(\sigma, \sigma^{-1})$, there exists the largest controllable closed R-shift inside $\mathscr{B}$ denoted by $\mathscr{B}_c$, that can be seen to have finite memory [14]. In our case there is a concrete way to characterize $\mathscr{B}_c$. Factor $M = FN$ with $F \in R^{l\times l}[u, u^{-1}]$ with $\det F \in S_b$ and $N \in R^{l\times q}[u, u^{-1}]$ left $(R[u, u^{-1}])^*$-factor prime. There exists $N' \in R^{(q-l)\times q}[u, u^{-1}]$ such that

$$\det \begin{bmatrix} N \\ N' \end{bmatrix} \in R_0(R[u, u^{-1}])^*. \tag{99}$$

Let $X \in R^{q\times l}[u, u^{-1}]$, $X' \in R^{q\times(q-l)}[u, u^{-1}]$ such that

$$[X\ X'] \begin{bmatrix} N \\ N' \end{bmatrix} = \begin{bmatrix} N \\ N' \end{bmatrix} [X\ X'] = rI_q, \tag{100}$$

where $r \in R_0$. Finally, factor $X' = PF'$ with $P \in R^{q\times(q-l)}[u, u^{-1}]$ primitive (i.e. right $\widetilde{R}_0$-factor prime) and $F' \in R^{(q-l)\times(q-l)}[u, u^{-1}]$ with $\det F' \in R_0$. We have the following:

Proposition 23.

$$\mathscr{B}_c = \operatorname{Im} P(\sigma, \sigma^{-1}). \tag{101}$$

Proof. '$\supseteq$': Since $0 = MX' = MPF'$, it follows that $MP = 0$. This proves the inclusion.

'$\subseteq$': By definition of $\mathscr{B}_c$ it is enough to show that if $P' \in R^{q\times s}[u, u^{-1}]$ is such that $\operatorname{Im} P'(\sigma, \sigma^{-1}) \subseteq \mathscr{B}$, then $\operatorname{Im} P'(\sigma, \sigma^{-1}) \subseteq \operatorname{Im} P(\sigma, \sigma^{-1})$. Now, if $MP' = 0$, it follows that $NP' = 0$ and so from (100) we have

$$rP' = XNP' + X'N'P' = X'N'P' = P(F'N'P'). \tag{102}$$

Since P is primitive, it follows that (see [9]) there exists $X \in R^{(q-l)\times s}[u,u^{-1}]$ such that $P' = PX$. This completes the proof. □

When $\operatorname{Ker} M(\sigma,\sigma^{-1})$ is not controllable, it is not possible to parametrize its elements through the image of a shift operator since the controllable part does not cover all of $\operatorname{Ker} M(\sigma,\sigma^{-1})$. We have to add to the controllable part a finitely generated free R-shift as shown in the next proposition.

Since $S_b^{-1}R[u,u^{-1}]$ is a PID, then there exists $M' \in R^{(q-l)\times q}[u,u^{-1}]$ such that

$$p := \det \begin{bmatrix} M \\ M' \end{bmatrix} \in S_b. \tag{103}$$

Let $X \in R^{q\times l}[u,u^{-1}]$, $X' \in R^{q\times(q-l)}[u,u^{-1}]$ be such that

$$\operatorname{Adj} \begin{bmatrix} M \\ M' \end{bmatrix} = [X\ X']. \tag{104}$$

We have the following:

Proposition 24. *Let* $\mathscr{B} := \operatorname{Ker} M(\sigma,\sigma^{-1})$. *Then we have the following*:

(1) *The controllable part of* $\mathscr{B}$ *is given by*

$$\mathscr{B}_c = \operatorname{Im} X'(\sigma,\sigma^{-1}). \tag{105}$$

(2) *The following decomposition*

$$\mathscr{B} = \tilde{\mathscr{B}} + \mathscr{B}_c \tag{106}$$

holds, where

$$\tilde{\mathscr{B}} = X(\sigma,\sigma^{-1})(\operatorname{Ker} p(\sigma,\sigma^{-1})I) \tag{107}$$

if finitely generated free as an R-module.

(3) *We have that*

$$\tilde{\mathscr{B}} \cap \mathscr{B}_c = (XM)(\sigma,\sigma^{-1})(\operatorname{Ker} p(\sigma,\sigma^{-1})I). \tag{108}$$

Proof. Suppose that $w = X(\sigma,\sigma^{-1})v$ with $p(\sigma,\sigma^{-1})v = 0$. Then $M(\sigma,\sigma^{-1})w = MX(\sigma,\sigma^{-1})v = 0$ and so

$$X(\sigma,\sigma^{-1})(\operatorname{Ker} p(\sigma,\sigma^{-1})I) \subseteq \operatorname{Ker} M(\sigma,\sigma^{-1}).$$

On the other hand it is clear that

$$\operatorname{Im} X'(\sigma,\sigma^{-1}) \subseteq \operatorname{Ker} M(\sigma,\sigma^{-1})$$

and thus

$$\mathscr{B} \supseteq \tilde{\mathscr{B}} + \operatorname{Im} X'(\sigma,\sigma^{-1}).$$

Suppose conversely that $M(\sigma,\sigma^{-1})w = 0$. Since the shift operator associated to the polynomial matrix $[X\ X']$ is surjective, there exists v such that

$$w = [X\ X'](\sigma,\sigma^{-1})v = X(\sigma,\sigma^{-1})v_1 + X'(\sigma,\sigma^{-1})v_2.$$

Moreover

$$p(\sigma,\sigma^{-1})v_1 = MX(\sigma,\sigma^{-1})v_1 = M[X\ X'](\sigma,\sigma^{-1})v = M(\sigma,\sigma^{-1})w = 0$$

and so we have that

$$\mathscr{B} \subseteq \tilde{\mathscr{B}} + \operatorname{Im} X'(\sigma,\sigma^{-1})$$

and thus

$$\mathscr{B} = \tilde{\mathscr{B}} + \operatorname{Im} X'(\sigma,\sigma^{-1}). \tag{109}$$

We want to show now that $\mathscr{B}_c = \operatorname{Im} X'(\sigma,\sigma^{-1})$. It is clear that $\mathscr{B}_c \supseteq \operatorname{Im} X'(\sigma,\sigma^{-1})$. Since $\mathscr{B}_c$ is closed and controllable, then by [14] there exists a polynomial matrix P such that $\mathscr{B}_c = \operatorname{Im} P(\sigma,\sigma^{-1})$. Let $w \in \mathscr{B}_c$. Then, since $p(\sigma,\sigma^{-1})$ is surjective, $w \in \operatorname{Im} pP(\sigma,\sigma^{-1})$ and so there exists $v \in \operatorname{Im} P(\sigma,\sigma^{-1})$ such that $w = p(\sigma,\sigma^{-1})v$. Using (109) we have that $v = v_1 + v_2$, where $v_1 \in \tilde{\mathscr{B}}$ and $v_2 \in \operatorname{Im} X'(\sigma,\sigma^{-1})$. Then

$$w = p(\sigma,\sigma^{-1})v = p(\sigma,\sigma^{-1})v_1 + p(\sigma,\sigma^{-1})v_2 = p(\sigma,\sigma^{-1})v_2 \in \operatorname{Im} X'(\sigma,\sigma^{-1}).$$

Finally, suppose that $w \in \tilde{\mathscr{B}} \cap \mathscr{B}_c$. Then $w = X(\sigma,\sigma^{-1})v_1 = X'(\sigma,\sigma^{-1})v_2$ with $p(\sigma,\sigma^{-1})v_1 = 0$. There exists u such that

$$\begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} = \begin{bmatrix} M \\ M' \end{bmatrix}(\sigma,\sigma^{-1})u.$$

This implies that $u \in \operatorname{Ker} p(\sigma,\sigma^{-1})$ and that $w = X(\sigma,\sigma^{-1})v_1 = XM(\sigma,\sigma^{-1})u$. Suppose conversely that $w = XM(\sigma,\sigma^{-1})u$, with $u \in \operatorname{Ker} p(\sigma,\sigma^{-1})$. Define $v := M(\sigma,\sigma^{-1})u$. Then it is clear that $p(\sigma,\sigma^{-1})v = 0$ and so $w \in X(\sigma,\sigma^{-1})(\operatorname{Ker} p(\sigma,\sigma^{-1}))$. Moreover, since

$$0 = p(\sigma,\sigma^{-1})u = XM(\sigma,\sigma^{-1})u + X'M'(\sigma,\sigma^{-1})u,$$

we have that $w = -X'M'(\sigma,\sigma^{-1})u$ and so $w \in \operatorname{Im} X'(\sigma,\sigma^{-1})$. □

The previous proposition provides a way for parametrizing the trajectories in $\operatorname{Ker} M(\sigma,\sigma^{-1})$ by an image representation and by an R-shift that can be generated through the techniques presented in the scalar case. The only drawback of this parametrization is that it is not injective since $\tilde{\mathscr{B}} \cap \mathscr{B}_c$ is not zero in general. Notice moreover that the previous proposition provides also an alternative way to compute the controllable part of $\operatorname{Ker} M(\sigma,\sigma^{-1})$ based on a completely different technique than the method shown in Proposition 23. It is worth noting that the method presented in Proposition 23 is usually more efficient and direct since it is based on algorithms working on

matrices with entries in the PID $R_0^{-1}R[u,u^{-1}]$, that are much easier than the analogos algorithms working on matrices with entries in the PID $S_b^{-1}R[u,u^{-1}]$.

We now turn to the non-homogeneous case

$$M(\sigma,\sigma^{-1})v = w. \tag{110}$$

If M is not R_0S_b-factor prime, there are all the difficulties which already appeared in the scalar case. Indeed, in the semisimple case, it follows from Theorem 21 that $\mathscr{B} := \operatorname{Im} M(\sigma,\sigma^{-1})$ is not closed and hence not easy to characterize. We concentrate here on the case when M is R_0S_b-factor prime. In this case we know that $\mathscr{B}$ is a closed controllable R-shift. It follows from the results in [5] that then $\mathscr{B}$ has necessarily finite memory and it will therefore admit, for the considerations done in the Introduction, a kernel representation. We now show how to find it explicitly.

We can factor $M = M_1M_2$ with $M_1 \in R^{l\times r}[u,u^{-1}]$ right R_0-factor prime and $M_2 \in R^{r\times q}[u,u^{-1}]$ left S_b-factor prime. We can therefore transform (110) into the system

$$M_1(\sigma,\sigma^{-1})x = w, \tag{111}$$

$$M_2(\sigma,\sigma^{-1})v = x. \tag{112}$$

Note that, by Theorem 18, (112) is always solvable in v for every x. Consider now Eq. (111). Note first that, by Theorem 15, $M_1(\sigma,\sigma^{-1})$ is injective. Hence, if a solution x exists, it is unique. It follows from Theorem 12 that there exists $Y \in R^{r\times l}[u,u^{-1}]$ and $r \in R_0$ such that $YM_1 = rI$. Hence, if x solves (111), we must have $rx = Y(\sigma,\sigma^{-1})w$. Therefore, we first find a necessary condition for $w \in \mathscr{B} := \operatorname{Im} M_1(\sigma,\sigma^{-1})$. We must have

$$r \mid Y(\sigma,\sigma^{-1})w. \tag{113}$$

Assume now that (113) is satisfied and consider $x = r^{-1}Y(\sigma,\sigma^{-1})w$. Substituting in (111), we obtain the second condition

$$(M_1Y - rI)(\sigma,\sigma^{-1})w = 0. \tag{114}$$

It is clear that the two conditions together (113) and (114) are necessary and sufficient for $w \in \mathscr{B}$. If we consider the shift operator

$$\psi\colon (R^l)^{\mathbb{Z}} \to (R^r/rR^r)^{\mathbb{Z}} \oplus (R^r)^{\mathbb{Z}}, \tag{115}$$

$$\psi(w) := \big(\pi(Y(\sigma,\sigma^{-1})w), (M_1Y - rI)(\sigma,\sigma^{-1})w\big), \tag{116}$$

where $\pi\colon (R^r)^{\mathbb{Z}} \to (R^r/rR^r)^{\mathbb{Z}}$ is the canonical projection. We have that

$$\operatorname{Ker}\psi = \mathscr{B} = \operatorname{Im} M(\sigma,\sigma^{-1}). \tag{117}$$

Moreover, if $w \in \operatorname{Im} M_1(\sigma,\sigma^{-1})$, the unique solution of (111) is given by $x = r^{-1}Y(\sigma,\sigma^{-1})w$. Note that we have also proved that $M_1(\sigma,\sigma^{-1})$ is open on its image so that also $M(\sigma,\sigma^{-1})$ is open on its image: equivalently we can say that (110) has the finite property as discussed in the Introduction.

What remains to be done is to show how to explicitly solve (112). We have already discussed how to solve the homogeneous equation associated to M_2. Hence, to solve (112), we only need to find an explicit solution. It follows from Theorem 18 that there exists $X \in R^{q\times r}[u,u^{-1}]$ and $r \in S_b$ such that $M_2X = rI$. Find first $\tilde{v} \in (R^r)^{\mathbb{Z}}$ such that $r(\sigma,\sigma^{-1})I\tilde{v} = x$. We have that

$$x = r(\sigma,\sigma^{-1})I\tilde{v} = M_2(\sigma,\sigma^{-1})X(\sigma,\sigma^{-1})\tilde{v} \tag{118}$$

which shows that $v = X(\sigma,\sigma^{-1})\tilde{v}$ is an explicit solution of (112). This completes our analysis.

We now present an example to show how to concretely apply our results.

Example 6. Let $R = \mathbb{R}[z]$. Let

$$M := \begin{pmatrix} -zu+2z^2 & -z(u-2) & 2z^2 \\ -u-1+3z & -2(u-2) & 3z \end{pmatrix} \tag{119}$$

and consider first the homogeneous difference equation

$$M(\sigma,\sigma^{-1})v = 0. \tag{120}$$

It is easy to check that the greatest common divisor of the principal minors is $z(u-2)$ so that M is left R_0S_b-factor prime. Using standard techniques we can factor $M = M_1M_2$, where

$$M_1 := \begin{pmatrix} z & z \\ 1 & 2 \end{pmatrix}, \qquad M_2 := \begin{pmatrix} -u+z+1 & 0 & z \\ z-1 & 2-u & z \end{pmatrix}. \tag{121}$$

M_2 is left S_b-factor prime. In studying the homogeneous problem we can forget M_1. Since $u-2$ is a common divisor of the principal minors of M_2, then by Theorem 22 we can argue that $\mathscr{B} := \operatorname{Ker} M(\sigma,\sigma^{-1}) = \operatorname{Ker} M_2(\sigma,\sigma^{-1})$ is not controllable.

Factor again M_2 as $M_2 = FN$, where

$$F := \begin{pmatrix} u & 1 \\ 2 & 1 \end{pmatrix}, \qquad N := \begin{pmatrix} -1 & 1 & 0 \\ z+1 & -u & z \end{pmatrix}. \tag{122}$$

Note that N is left $R[u,u^{-1}]^*$-factor prime. Complete it to

$$\tilde{N} := \begin{pmatrix} -1 & 1 & 0 \\ z+1 & -u & z \\ 1 & 0 & 0 \end{pmatrix}. \tag{123}$$

We have that

$$\operatorname{Adj}(\tilde{N}) = \begin{pmatrix} 0 & 0 & z \\ z & 0 & z \\ u & 1 & u-z-1 \end{pmatrix}. \tag{124}$$

Hence the controllable part of $\mathscr{B}$ is

$$\mathscr{B}_c = \operatorname{Im} \begin{pmatrix} z \\ z \\ \sigma - (z+1) \end{pmatrix}. \tag{125}$$

Complete now M_2 to

$$\tilde{M}_2 := \begin{pmatrix} -u+z+1 & 0 & z \\ z-1 & 2-u & z \\ 0 & 1 & 1 \end{pmatrix}. \tag{126}$$

We have that

$$\operatorname{Adj}(\tilde{M}_2) = \begin{pmatrix} 2-u-z & z & z(u-2) \\ 1-z & -u+z+1 & z(u-2) \\ z-1 & u-z-1 & (u-2)(u-z-1) \end{pmatrix} \tag{127}$$

and $q := \det \tilde{M}_2 = (u-2)(u-1) \in S_b$. Let

$$X := \begin{pmatrix} 2-u-z & z \\ 1-z & -u+z+1 \\ z-1 & u-z-1 \end{pmatrix}, \qquad X' := \begin{pmatrix} z(u-2) \\ z(u-2) \\ (u-2)(u-z-1) \end{pmatrix}. \tag{128}$$

These matrices are such that $\operatorname{Adj}(\tilde{M}_2) = [X\ X']$. Then the controllable part of $\mathscr{B}$ is given also by $\operatorname{Im} X'(\sigma, \sigma^{-1})$ (as it can be easily verified) and moreover

$$\mathscr{B} = \tilde{\mathscr{B}} + \mathscr{B}_c, \tag{129}$$

where $\tilde{\mathscr{B}} \simeq X(\sigma, \sigma^{-1})(\operatorname{Ker} q(\sigma, \sigma^{-1})I)$. Notice that $w \in \operatorname{Ker} q(\sigma, \sigma^{-1})I$ if and only if

$$w^{\mathrm{T}}(t) = CA^t B, \quad \forall t \in \mathbb{Z}, \tag{130}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}, \qquad C = (1\ 0)$$

and B may be any matrix in $R^{2\times 2}$. Noting that $X = X_0 + X_1 u$, where

$$X_0 := \begin{pmatrix} 2-z & z \\ 1-z & z+1 \\ z-1 & -z-1 \end{pmatrix}, \qquad X_1 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}, \tag{131}$$

we have that $v \in X(\sigma, \sigma^{-1})(\operatorname{Ker} q(\sigma, \sigma^{-1})I)$ if and only if

$$v^{\mathrm{T}}(t) = CA^t(BX_0^{\mathrm{T}} + ABX_1^{\mathrm{T}}), \quad \forall t \in \mathbb{Z}, \tag{132}$$

where B is free to vary in $R^{2\times 2}$.

We pass now to consider the non-homogeneous equation

$$M(\sigma, \sigma^{-1})v = w. \tag{133}$$

Consider the factorization $M = M_1 M_2$ as above. In this case we can immediately see that

$$\operatorname{Im} M(\sigma, \sigma^{-1}) = \operatorname{Im} M_1(\sigma, \sigma^{-1}) := \left\{ \begin{pmatrix} zw_1 \\ w_2 \end{pmatrix} \mid w_1, w_2 \in R^{\mathbb{Z}} \right\}. \tag{134}$$

Let now fix

$$w = \begin{pmatrix} zw_1 \\ w_2 \end{pmatrix}. \tag{135}$$

The unique solution of $M_1(\sigma, \sigma^{-1})x = w$ is given by

$$x = \begin{pmatrix} 2w_1 - w_2 \\ -w_1 + w_2 \end{pmatrix}. \tag{136}$$

To find a particular solution of (133), solve first the equation

$$(1-u)(2-u)I\tilde{v} = \begin{pmatrix} 2w_1 - w_2 \\ -w_1 + w_2 \end{pmatrix}. \tag{137}$$

A particular solution is given by

$$\tilde{v}(t) := \sum_{k=0}^{t-1} \begin{pmatrix} 2w_1(t) - w_2(t) & 0 \\ -w_1(t) + w_2(t) & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}^{t-1-k} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{138}$$

and so a particular solution of $M_2(\sigma, \sigma^{-1})v = x$ is then given by $X(\sigma, \sigma^{-1})\tilde{v}$ which provides a particular solution also of (133).

3.4. *An extension to the ring of local analytic functions*

Even if we consider only one variable, the ring of locally convergent power series $R = k\{z\}$ ($k = \mathbb{R}, \mathbb{C}$) is not a PID. However, some parts of the theory developed in this section can still be applied to this case. Two basic facts are the key ingredients to do this: first, its completion $\hat{R} = k[[z]]$ is indeed a PID; second we have the important result by Artin [1] already recalled in Section 1 which essentially permits us to solve systems of polynomial equations in R once we know that a solution exists in $\hat{R}$. Using these two ingredients in a carefully way, we can prove in a lengthy but straightforward way that Theorem 12 and Proposition 14 still hold true if $R = k\{z\}$ and $S = S_b, \widetilde{S}_b, R_0$. With this we can then show that all the results for systems of difference equations established in the last two subsections can be extended to the case $R = k\{z\}$. Notice also that Proposition 11 admits a straightforward extension to the matrix case. Finally, notice that for the ring $\hat{R} = k[[z]]$ also non-homogeneous problems with matrices M which are not necessarily $\hat{R}_0 S_b$-factor prime can be treated via the results obtained in Section 2 for the complete PID's. This permits us, in principle, to study general matrix non-homogeneous problems for $R = k\{z\}$ using the same techniques than in the scalar case as in Example 3 of Section 2. We omit all details.

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